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A Relativistic Electron Interaction Investigation in an Electromagnetic Field

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Abstract: A charged particle in an electromagnetic field is discussed using the Dirac equation of the free electron. The time-dependent components of the spin expectation value are calculated. The results are interesting from the educational point of view as they modify the views often expressed in the literature concerning the direction quantization of the spin vector. As a second step, the Maxwell equations of electrodynamics are extended into 4-dimensional Minkovsky space. Two special cases together with the most general case of N electrons in an electromagnetic field are discussed in detail and the relativistic limit $v \rightarrow c$ is numerically evaluated. On the other hand, for $v \ll c$ the well-known classical results are reproduced.

1. Introduction

This work discusses a relativistic electron in an electromagnetic field and one way to study this is via a relativistic expansion of the Maxwell equations. The dynamics of charged particles in an electromagnetic field remain of interest due to both relativistic and quantum effects. While relativistic effects are covered by the Dirac equation, quantum effects are modeled by the Schrodinger equation for free electrons in a magnetic field; the corresponding work by L D Landau is discussed later in this section.

It is well known that the Maxwell equations help study the mutual interactions between electric and magnetic fields [1], which has drawn many researchers into the field. To derive the Maxwell equations, Mohr [2], discussed the analogy of the six-component electromagnetic field solutions of the Maxwell equations matrix to that of the four-component Dirac equation solutions. Furthermore, Li et al. [3] established the Dirac equation in terms of quaternions with the aim of unifying electrons, that is, matter waves and electromagnetic waves. Based on this idea, the authors were able to derive the electrons' Maxwell equations, referred to as electronic Maxwell equations (EME). The EME theory is conveniently used to explore statistical knowledge about the election, for example, its probability density. It was also established that EME is helpful to control the phase shift of an electron beam without the necessity of dealing with quantum operators. In [4, 5], the Maxwell equations were studied as a quantum equation for one photon. The study highlighted that time-dependent Maxwell equations for Faraday's and Ampere's law could be presented as a three-component equation and showed how the Maxwell equations could be derived from first principles simultaneously, like the derivation of the Dirac relativistic equation for an electron. Furthermore, Maxwell equations can be observed as "one-photon spin-one quantum equations" and it was suggested that the Maxwell equations could be used for quantum theory interpretation guidelines. Kar et al. [6] proposed the calculation of a photon energy loss propagated in a nonzero conductivity coefficient vacuum

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by applying the Maxwell equations. Reggia [7] improved the symmetry of Maxwell equations to cater to complex-valued electromagnetic fields by generalizing the equations. The use of generalized equations allowed for a consistent experimental prediction as the electromagnetic wave state in a particular medium can be described in this way. In [8, 9], the mass of a photon was theoretically investigated using both Maxwell equations and the theory of Compton scattering. The findings were that the photon energies are small and depend on the medium’s electromagnetic properties and the frequency of the wave. Maxwell equations in nonlinear optics were theoretically studied in [10]. It was also stated that in a vacuum, Maxwell equations facilitate the creation of linear electromagnetic fields and that an approximation of these equations generates nonlinear effects of electromagnetic fields. The study done in [11] showed the Maxwell equations can be written in terms of Hamilton’s Quaternions based on the Quaternion Axiom. This study postulated that the physical space can be regarded as a quaternion structure. More interestingly, Fialová and Pochylý [12], presented Maxwell equations in vector and scalar variants that allowed the use of Gauss’ theorem appropriate for both electrical and magnetic inductions. With this new methodology, the effects of non-stationary boundary conditions on electromagnetic fields in continuous media can be analyzed.

When considering an electron in an electromagnetic field it is shown by the Dirac theory that its energy eigenvalues are twofold degenerate which is explained by the two discrete directions of the electron spin with respect to the z-axis. The electron spin is thus a relativistically explained particle property. Landau [13] presented a quantum mechanical description of a free electron gas in a magnetic field. The corresponding energy eigenvalues are quantized and degenerate (Landau levels). Due to the electron spin the energy levels split into two Zeeman terms in the presence of an external magnetic field. The theory thus explains the phenomenon of diamagnetism in metals.

The article is organized as follows. In Section 2 the time-dependent spin expectation values of the electron in a magnetic field are calculated and the phenomenon of direction quantization of the spin vector is revisited. Even though the topic of direction quantization is extensively discussed in the textbook literature, the question of time dependence of the spin vector is frequently overlooked. In Section 3 numerical results for a charged particle in an electromagnetic field are presented and several special cases are discussed. Conclusions are drawn in Section 4.

2. Theory

The formulation of Maxwell equations in 4-dimensional Minkovsky space is frequently discussed in the literature, see e.g. ref [14]. This covariant formulation of classical electrodynamics proves that the Maxwell equations are relativistically invariant. The theory also shows how electric and magnetic fields are altered under Lorentz transformation from one inertial reference frame to another. Due to its importance in physics research, this discussion is ongoing as documented in [15-18]. In this article, we thus concentrate on particular solutions to the general problem. The relativistic Lorentz force is determined from

$$\frac{d\boldsymbol{p}}{d\tau} = q\,\gamma\,\left(E + v \times B, \frac{i}{c}\, E \cdot u\right) = \gamma\,\left(F, \frac{i}{c}\, F \cdot u\right)$$

$$F = q\,(E + u \times B) \qquad (1)$$

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Note that in cases where the Lorentz force $\boldsymbol{F}$ is perpendicular to the particle velocity $\boldsymbol{u}$ or in the case of a force-free particle, the momentum is conserved.

To investigate the electron spin further, an electron in a magnetic field $\boldsymbol{B} = B\,\boldsymbol{e_z}$ is considered. The x, y-components of the time-dependent spin expectation value are described by the differential equation

$$\frac{d^2}{d\,t^2}\ \langle S_i \rangle\,(t) + \omega^2\ \langle S_i \rangle\,(t) = 0 \tag{2}$$

$i = x, y$ with the well-known solutions

$$(S_x)\,(t) = A\ \sin(\omega\,t)\ + B\ \cos(\omega\,t)$$

$$\langle S_y \rangle\ (t) = \ -B\ \sin(\omega\,t)\ + A\ \cos(\omega\,t) \tag{3}$$

The constants A, B are determined from the initial conditions so that

$$A = \ \langle S_y \rangle\ (t = 0)$$

$$B = \ \langle S_x \rangle\,(t = 0) \tag{4}$$

Note the interdependence of the two solutions of Eq. (3) as

$$\frac{d}{dt}\ \langle S_x \rangle\,(t) = \ \omega\ \langle S_y \rangle\,(t)$$

$$\frac{d}{dt}\ \langle S_y \rangle\,(t) = \ -\,\omega\,\langle S_x \rangle\,(t) \tag{5}$$

On the other hand, the z-component of the spin expectation value is time-independent with

$$\frac{d}{dt}\ \langle S_z \rangle\,(t) = 0$$

$$\langle S_z \rangle\,(t) = \ \langle S_z \rangle\,(t = 0) = \pm\,\frac{1}{2}\ \hbar \tag{6}$$

The tip of the spin vector is described as a function of time t a circle in the xy-plane with a radius

$$r = \ \sqrt{\langle S_x \rangle^2 + \langle S_y \rangle^2} \tag{7}$$

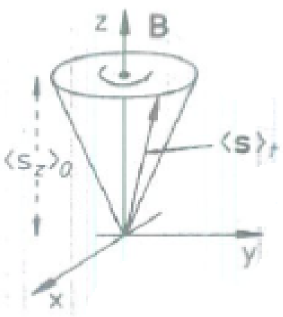

Figure 1: Precession of the expectation value $\langle S \rangle$ around the direction of the magnetic field B. Adopted from reference [19].

Even though both $\langle S_x \rangle (t)$ and $\langle S_y \rangle (t)$ are according to Eq. (3) above time-dependent, the radius r and the angle

$$\alpha = tan^{-1} \frac{\sqrt{\langle S_x \rangle^2 + \langle S_y \rangle^2}}{\langle S_z \rangle} \tag{8}$$

of the precession cone are independent of time t. As the angle α is via

$$\alpha = \omega\, t = 2\, \frac{\mu_B}{\hbar}\, B\, t = const \tag{9}$$

related to the applied magnetic field, it follows that the larger the magnetic field B the shorter the period T necessary for one revolution. Also note that ω is twice the Larmor frequency of an electron spin

$$\omega = 2\, \omega_L \sim 10^{11}\ s^{-1}$$

$$T = \frac{2\, \pi}{\omega}\ \sim\ 10^{-11}\ s \tag{10}$$

The above results also show that the picture often described in the literature about the electron spin being fixed in space with its projection either parallel or antiparallel to the z-axis (direction quantization) is not entirely correct. Such a scenario would also contradict the fundamental commutator relation

$$\left[S_x\,, S_y\right]_- = i\, \hbar\, S_z \tag{11}$$

which says that two components of the spin operator can never be simultaneously accurately measured.

3. Results

In this section, the following special cases are discussed. Numerical results were plotted using the graphing utility gnuplot 5.0 for Linux.

3.1. A charged particle in a constant electric field and vanishing magnetic field.

In this case

$$m\, \gamma\, u = q\, E\, t \tag{12}$$

$$u = \frac{q\, E\, t}{m\, \gamma} = \frac{q\, E\, t}{\sqrt{m^2 + \left(\frac{q\, E\, t}{c}\right)^2}} \tag{13}$$

Corresponding numerical results are plotted in Figure 2 below.

Plotted is the particle speed ${}^{u}/{}_{c}$ as a function of time t for a charged particle in a constant electric field E. For small t

$$u\, (t) = \frac{q}{m}\, E\, t \tag{14}$$

increases linearly with t and this part of the curve agrees with the corresponding classical results. On the other hand, for large t the asymptotic value

$$\lim_{t \to \infty} u(t) = c$$

is reproduced. The particle speed approaches the speed of light.

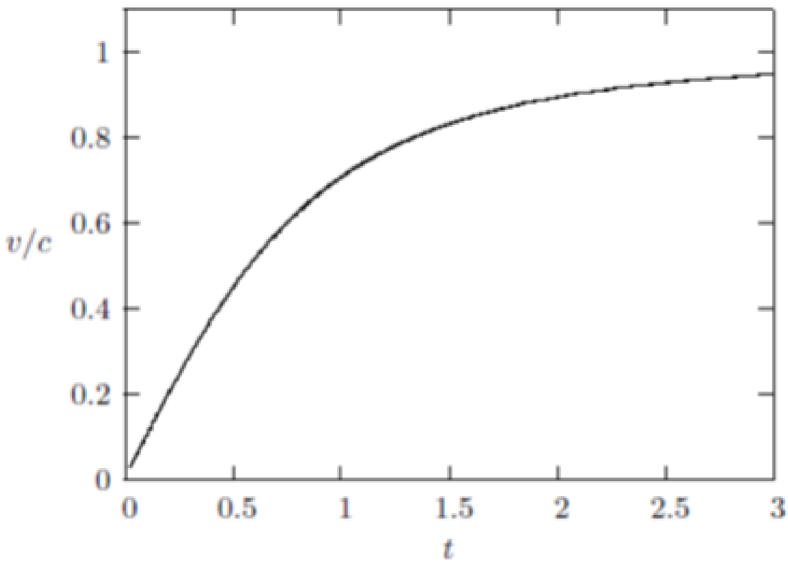

Figure 2: Relative particle speed $^u/_c$ as a function of time t.

The results of Fig 2 qualitatively agree with those of reference [20] where it is pointed out that at large accelerating voltages U the relativistic particle speed approaches that of the speed of light with a $^v/_c$ ratio of 0.941 being achieved for accelerating voltages $U \sim 10^6\ eV$. This is consistent with the relationship

$$q\,U = \ (\gamma - 1)\, m\, c^2$$

between accelerating voltage and the relativistic kinetic energy.

3.2. A charged particle in a constant magnetic field

Here the charged particle's path in the magnetic field is a circle with a radius r. One can show that this radius is proportional to the particle's relativistic momentum, that is,

$$r = \frac{p}{q\,B} = \frac{m}{q\,B}\ \frac{u}{\sqrt{1 - {u^2}/{c^2}}} \tag{15}$$

Corresponding results are plotted in Figure 3 below.

Note that for $v \ll c$ the radius $r \sim v$ as in the classical non-relativistic case. However, for $v \to c$ $r \sim p$ the relativistic momentum. Eq (15) agrees with the results of reference [21] for the ultra-relativistic limit. Here both r and the cyclotron frequency ω_c depend on the particle energy. This fact is utilized in synchrotrons where the magnetic field is increased with the particle momentum p so that both r and ω_c remain constant. In this way, charged particles can be accelerated to energies of the order of $10^9\ eV$ [22].

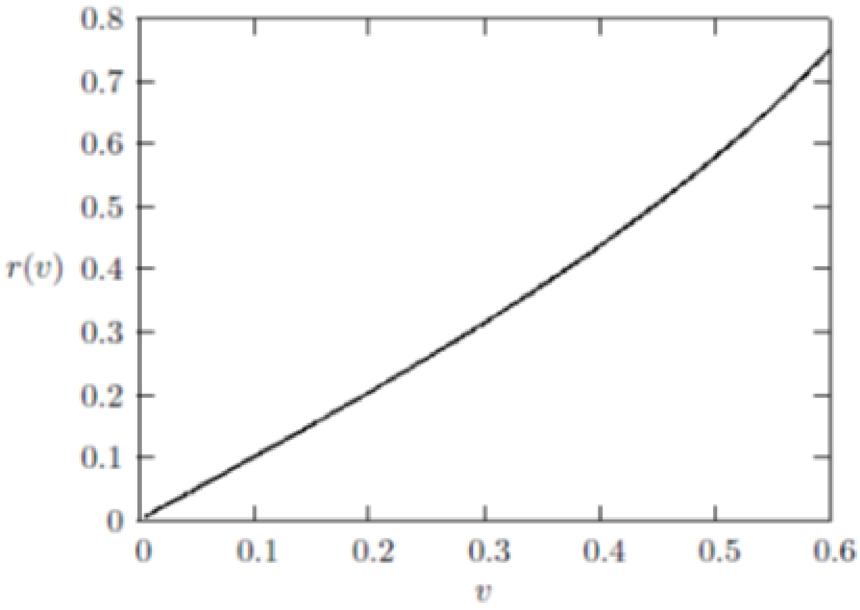

Figure 3: Radius r as a function of particle speed u.

3.2. System of N charged particles in an electromagnetic field occupying the volume V.

If the force exerted by the field on the boundary element $d\boldsymbol{A}$ is

$$\boldsymbol{F} = \rho\,\boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B} \tag{16}$$

then for particular directions of the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ the angle φ enclosed by the two vectors $d\boldsymbol{F}$ and $d\boldsymbol{A}$ can be calculated as a function of the relative particle velocity $f = {}^{v}/_{c}$.

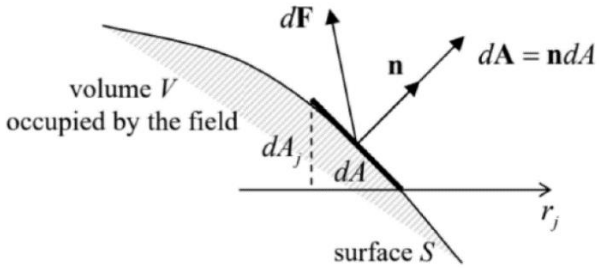

Figure 4: Force dF exerted on boundary element dA of the volume V occupied by the field. Adopted from reference [21].

The angle φ is calculated from

$$\varphi = cos^{-1}\left(\frac{1}{\sqrt{1+\left(\frac{1}{f}\right)^2}}\right) \tag{17}$$

and corresponding results are plotted in Figure 5 below.

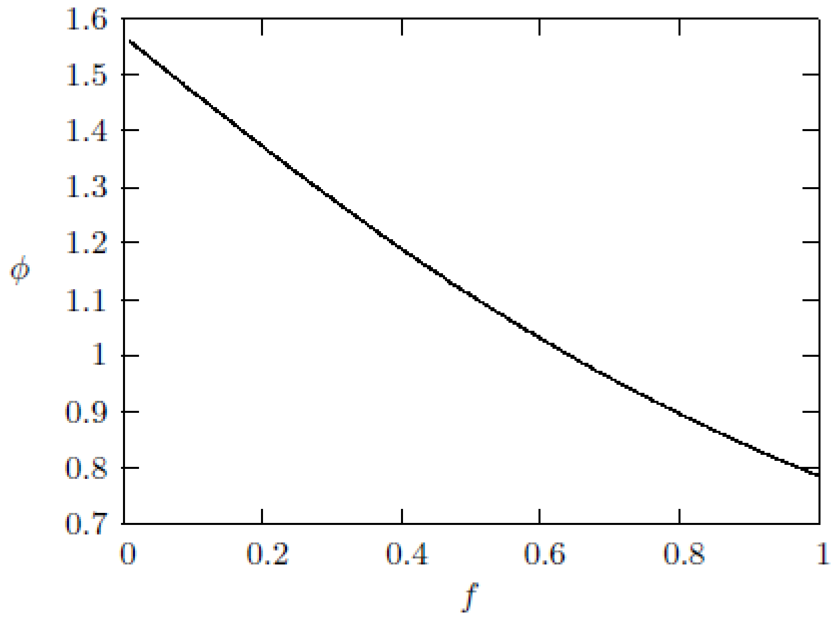

Figure 5: Angle φ as a function of the relative particle speed $f = \frac{v}{c}$.

The vector $\boldsymbol{F}$ generally lies in the xy-plane but becomes parallel to the $\boldsymbol{E}$-field in the non-relativistic limit. For $v \ll c$ $\varphi = \frac{\pi}{2}$ and $d\boldsymbol{F}$ is perpendicular to $d\boldsymbol{A}$. With increasing particle velocity v the angle φ is continuously reduced and finally becomes equal to $\frac{\pi}{4}$ at $v \cong c$.

4. Conclusion

In this study, a relativistic electron in an electromagnetic field was investigated. The direction quantization of the spin vector was revisited. It is shown that the angle α of the precession cone is for a given spin vector $\boldsymbol{S} = (S_x, S_y, S_z)$ independent of the magnetic field B. It then follows that the corresponding period $T \sim \frac{1}{B}$ meaning the larger the magnetic field the smaller the period T becomes. Furthermore, the precession of the spin vector is in agreement with the fundamental commutator relations for the cartesian components of $\boldsymbol{S}$.

In the case of charged particles in constant electric and magnetic fields it was shown that in both the ultra-relativistic and the non-relativistic limit our results agree with those of other authors. One can therefore assume that our data are also correct in the intermediate range of particle velocities.

In the most general case of N electrons in an electromagnetic field the angle φ of the force vector with respect to the boundary element $d\boldsymbol{A}$ was calculated for arbitrary particle velocities v. With decreasing particle velocity the magnetic contribution to the force $\boldsymbol{F}$ of Eq (16) decreases and the angle φ enclosed by $d\boldsymbol{F}$ and $d\boldsymbol{A}$ increases. Only in the non-relativistic case $v \ll c$ does $d\boldsymbol{F}$ turn out to be perpendicular to $d\boldsymbol{A}$.

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